$$=2x^2(1-x)^{-2}+x(1-x)^{-1}=x(1+x)(1-x)^{-2}=\frac{n^p+1}{(n^p-1)^2}$$

where we must have $|x|<1$.

Also solved by Henry Heaton, A. H. Holmes, and G. B. M. Zerr.

CALCULUS.

217. Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, Ohio.

Find $\lim_{n\doteq\infty}\frac{1}{n}\sqrt[n]{[(n+1)(n+2)\ldots\ldots(2n)]}$.

I. Solution by the PROPOSER.

Let $x=\lim_{n\doteq\infty}\frac{1}{n}\sqrt[n]{[(n+1)(n+2)\ldots\ldots 2n]}$

$$=\lim_{n\doteq\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)}.$$

Then $\log x=\lim_{n\doteq\infty}\frac{1}{n}\log\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)\right]$

$$=\lim_{n\doteq\infty}\frac{1}{n}\sum_{\lambda=1}^{\lambda=n}\log\left(1+\frac{\lambda}{n}\right)=\lim_{n\doteq\infty}\frac{1}{n}\sum_{\lambda=1}^{\lambda=n}\left(\frac{\lambda}{n}-\frac{\lambda^2}{2n^2}+\frac{\lambda^3}{3n^3}-\ldots\ldots\right)$$

$$=\lim_{n\doteq\infty}\sum_{\lambda=1}^{\lambda=n}\sum_{\kappa=1}^{\kappa=n}(-1)^{\kappa-1}\frac{\lambda^\kappa}{\kappa n^{\kappa+1}}.$$

If the method of differences is used for $\sum_{\lambda=1}^{\lambda=n}\lambda^\kappa=1^\kappa+2^\kappa+3^\kappa+\ldots\ldots$, the κth series of differences is

$$(\kappa+1)^\kappa-\binom{\kappa}{1}\kappa^\kappa+\binom{\kappa}{2}(\kappa-1)^\kappa-\binom{\kappa}{3}(\kappa-2)^\kappa+\ldots$$
$$+(-1)^{\kappa-1}\binom{\kappa}{\kappa-1}2^\kappa+(-1)^\kappa 1^\kappa\equiv\kappa!.$$

The $(\kappa+1)$th series is

$$(\kappa+2)^\kappa-\binom{\kappa+1}{1}(\kappa+1)^\kappa+\binom{\kappa+1}{2}\kappa^\kappa-\ldots+(-1)^\kappa\binom{\kappa+1}{\kappa}2^\kappa+(-1)^{\kappa+1}1^\kappa\equiv 0,$$

κ being a positive integer.

If the first given number is represented by a and the successive differences by $d_1, d_2, \ldots\ldots$

$$S_{n,\kappa}=\binom{n}{1}a+\binom{n}{2}d_1+\binom{n}{3}d_2+\ldots+\binom{n}{\kappa+1}d_\kappa.$$

First $\lim_{n \doteq \infty} \frac{1}{\kappa n^{\kappa+1}} S_{n,\kappa}$ must be sought.

Only in the last term of $S_{n,\kappa}$ n appears in $\kappa+1$ factors, therefore the preceding terms disappear, and

$$\lim_{n \doteq \infty} \frac{1}{\kappa n^{\kappa+1}} S_{n,\kappa} = \lim_{n \doteq \infty} \frac{1}{\kappa n^{\kappa+1}} \left[\frac{n(n-1)(n-2).....(n-\kappa)}{(\kappa+1)!} \kappa! \right] = \frac{1}{\kappa(\kappa+1)}.$$

Therefore $\lim_{n \doteq \infty} \sum_{\lambda=1}^{\lambda=n} \sum_{\kappa=1}^{\kappa=n} (-1)^{\kappa-1} \frac{\lambda^\kappa}{\kappa n^{\kappa+1}} = \sum_{\kappa=1}^{\kappa=\infty} (-1)^{\kappa-1} \frac{1}{\kappa(\kappa+1)}$

$$= \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + ... = (1 - \frac{1}{2}) - (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) - .....$$

$$= 2(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...) - 1 = 2 \log 2 - 1 = \log \frac{4}{e}.$$

Hence, $\log x = \log \frac{4}{e}$; and $x = \lim_{n \doteq \infty} \frac{1}{n} \sqrt[n]{[(n+1)(n+2) .... 2n]} = \frac{4}{e}$.

II. Solution by S. A. COREY, Hiteman, Iowa.

Evidently, $\frac{1}{n} \sqrt[n]{[(n+1)(n+2) .... (2n)]}$

$$= \frac{1}{n} \sqrt[n]{n^n (1+\frac{1}{n})(1+\frac{2}{n}) .... (2)}$$

$$= \sqrt[n]{(1+\frac{1}{n})(1+\frac{2}{n}) ....(2)} = s \text{ (say)}.$$

Therefore, $\log s = \frac{1}{n} [\log (1+\frac{1}{n}) + \log(1+\frac{2}{n}) + ... + \log 2]$.

Letting $dx = 1/n$, we have,

$$\lim_{n \doteq \infty} \log s = \int_1^2 \log x \, dx = 2 \log 2 - 1, \text{ or } s = \frac{4}{e}.$$

Also solved by Henry Heaton, and J. Scheffer. Several incorrect solutions were received.

209. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A thread makes $n(=30)$ equidistant spiral turns around a rough cone whose altitude is $h(=10$ feet), and radius of base $r(=11$ inches). How far will a bird fly in unwinding the thread if the part unwound is at all times perpendicular to the axis of the cone?